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Steady penetration of a rigid cone into pressure-dependent plastic material

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Abstract

The objective of the present paper is to find a semi-analytical axisymmetric solution for steady penetration of a rigid cone into pressure-dependent plastic material obeying the double-shearing model. As expected, the solution is singular near the maximum friction surface. It is important to mention that the singularity is not due to the geometry of the problem but the friction law. The type of the singularity is the same as in plane-strain solutions based on the double-shearing model and in classical plasticity. This allows for calculating the strain rate intensity factor. The solution is illustrated by a numerical example.

Keywords: Penetration; Friction; Singular solution; Double-shearing model; Pressure-dependent plasticity

1. Introduction

Steady penetration of a rigid cone or wedge into a medium belongs to a group of classical plane-strain and axisymmetric problems in plasticity that also includes compression between parallel plates, flow through channels and other similar problems. The main assumption accepted in all these solutions is that the orientation of the principal stress depends on a single coordinate only. Solutions for plane-strain and axisymmetric penetration into various plastic media have been obtained by Fleck and Durban (1991), Durban and Rand (1991), Durban and Fleck (1992) and Durban (1999). These studies have focused on

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the singular behavior of solutions in the vicinity of the apex. In the present paper steady penetration of a rigid cone with the rough surface into rigid/plastic material obeying the double shearing model of pressure-dependent plasticity is considered. A solution to an analogous plane-strain problem has been obtained by Alexandrov and Lyamina (in press). The double-shearing model for granular materials under plane-strain conditions has been proposed by Spencer (1964). The systems of equations for axisymmetric and three-dimensional deformations have been given in Spencer (1982). The model is based on the Coulomb–Mohr yield condition and the assumption that deformation occurs by shear on the characteristic curves of stress equations consisting of the yield condition and the equilibrium equations. It does not include the normality rule but includes the incompressibility equation. Another important property of the model is that the stress characteristics coincide with the velocity characteristics. A number of analytical and semi-analytical plane-strain solutions based on the double-shearing model generalizing the corresponding solutions in classical plasticity have been obtained by Pemberton (1965), Marshall (1967), Spencer (1982), Alexandrov and Lyamina (2003). All of these solutions lead to singular velocity fields in the vicinity of friction surfaces where the maximum friction law is adopted. It is important to note that this type of singularity is quite different from that emphasized in Fleck and Durban (1991), Durban and Rand (1991), Durban and Fleck (1992), Durban (1999) and Papanastasiou et al. (2003). The latter is caused by the geometry of the problem and occurs in the vicinity of the apex, whereas the former is caused by the maximum friction law and occurs in the vicinity of the friction surface. The maximum friction law postulates that a characteristic direction (in the case of the double shearing model the characteristic directions for stress and velocity equations coincide) is tangent to the friction surface. In the case of plane-strain deformation, it has been shown in Alexandrov and Lyamina (2002) that the singular solutions occur near friction surfaces where an envelope of characteristics coincides with such a surface. There is no general result on singular solutions for axisymmetric flows of materials obeying the double-shearing model. However, a solution for flow through an infinite converging channel shortly described in Spencer (1982) shows that this particular velocity field is singular. Other solutions for axisymmetric deformation of materials obeying the double-shearing model are given in Spencer (1983, 1984, 1986). However, these solutions do not involve the maximum friction law. In particular, in Spencer (1984) steady penetration of a rigid cone with a frictionless wall has been studied. In the present paper, the same problem with friction is solved, assuming the maximum friction law at the cone surface. An essential difference between these formulations is that in Spencer (1984) a face regime on the yield surface occurs whereas the present solution requires an edge regime. The latter solution shows that the velocity field is singular near the maximum friction surface and the type of singularity is the same as in plane-strain solutions based on the double-shearing model (Alexandrov and Lyamina, 2002) and in arbitrary flows of classical plasticity (Alexandrov and Richmond, 2001).

2. Statement of the problem

A rigid cone is penetrating an incompressible pressure-dependent plastic solid under axisymmetric conditions. End effects are neglected. Without loss of generality, it is possible to assume that the cone is motionless whereas the material moves with a velocity U as shown in Fig. 1. It is convenient to introduce a spherical coordinate system $r\theta\varphi$ with its origin at the cone apex. Then, the surface of the cone is defined by the equation $\theta = \theta_0$. It is assumed that there exists a rigid/plastic boundary defined by the equation $\theta = \theta_p$. The value of θ_p should be found from the solution. The solid obeys the double-shearing model (Spencer, 1982). The model includes the Coulomb–Mohr yield condition. In the case of axisymmetric deformation, several regimes on this yield condition described in Spencer (1982) are possible. For the problem under consideration, the appropriate regime corresponds to point F (Fig. 2) and is defined by the following equations:

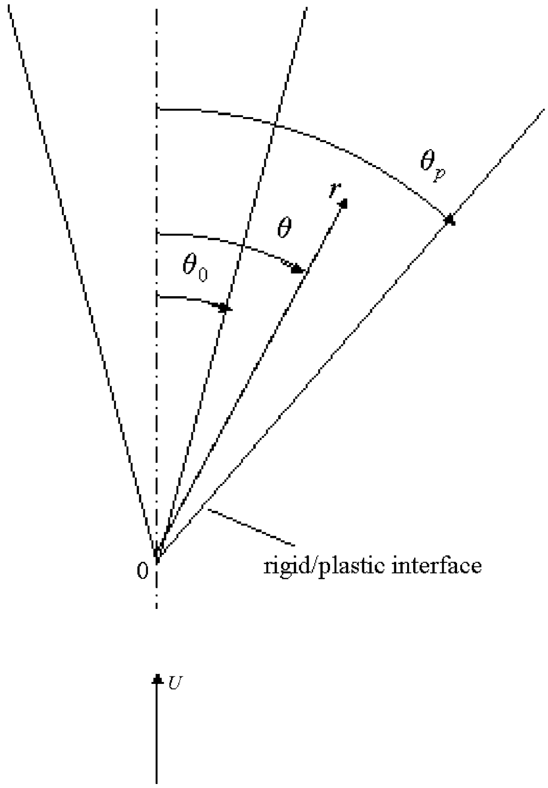

Fig. 1. Notation for steady penetration by a rigid cone.

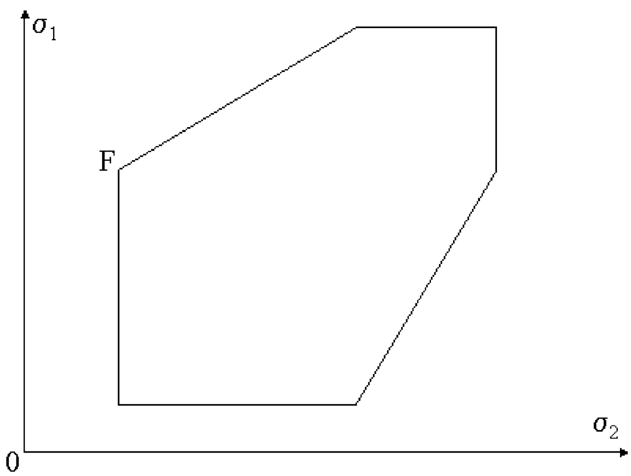

Fig. 2. Coulomb–Mohr yield hexagon for axially symmetric stress state. σ_1 and σ_2 are the principal stresses in a meridian plane.

$$(\sigma_{rr} + \sigma_{\theta\theta})\sin\phi + \sqrt{(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2} = 2c\cos\varphi$$
$$2\sigma_{\varphi\varphi} = \sigma_{rr} + \sigma_{\theta\theta} + \sqrt{(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2} \tag{1}$$

where σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{\varphi\varphi}$ and $\sigma_{r\theta}$ are the components of the stress tensor in the spherical coordinate system, c is the cohesion and ϕ is the angle of internal friction. To obtain the closed form system for stress, Eqs. (1) should be complemented with the equilibrium equations

$$r\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{r\theta}}{\partial\theta} + 2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} + \sigma_{r\theta}\cot\theta = 0$$
$$r\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\cot\theta = 0 \tag{2}$$

The velocity equations given in Spencer (1982) can be rewritten in the spherical coordinates in the form

$$r\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial\theta} + 2u_r + u_\theta\cot\theta = 0 \tag{3}$$

$$(\cos 2\psi + \sin\phi)\frac{\partial u_r}{\partial\theta} + r(\cos 2\psi - \sin\phi)\frac{\partial u_\theta}{\partial r} - (\cos 2\psi + \sin\phi)u_\theta - \left(r\frac{\partial u_r}{\partial r} - \frac{\partial u_\theta}{\partial\theta} - u_r\right)\sin 2\psi$$
$$+ 2\sin\phi u_\theta\frac{d\psi}{d\theta} = 0 \tag{4}$$

Here (3) is the incompressibility equation and ψ is the angle which the axis corresponding to the maximum principal stress in a meridian plane makes with the r-direction. Since the flow is steady ψ is independent of the time. Also, by assumption, ψ is independent of r. These facts have been taken into account to derive Eq. (4) from the general equation given in Spencer (1982). Eq. (4) ensures that the deformation occurs by two simultaneous superimposed shearing deformations on the characteristics of the stress equations.

The surface of the cone is rough, and the maximum friction law is adopted there. In the case under consideration, this law reads

$$\psi = \psi_w = \frac{\pi}{4} + \frac{\phi}{2} \tag{5}$$

at $\theta = \theta_0$. Eq. (5) states that the friction surface (line in a meridian plane) coincides with a characteristic direction of the system of Eqs. (1) and (2), and (3) and (4).

The velocity boundary condition on the cone surface, $\theta = \theta_0$, is

$$u_\theta = 0 \tag{6}$$

The velocity field is assumed to be continuous across the rigid/plastic boundary, which is not a characteristic. Therefore,

$$u_r = U\cos\theta_p \quad \text{and} \quad u_\theta = -U\sin\theta_p \tag{7}$$

at $\theta = \theta_p$.

3. Stress solution

With the use of the standard substitution

$$\sigma_{rr} = -p + q\cos 2\psi, \quad \sigma_{\theta\theta} = -p - q\cos 2\psi, \quad \text{and} \quad \sigma_{r\theta} = q\sin 2\psi \tag{8}$$

where

$$p = -\frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta}), \quad q = \frac{1}{2}\sqrt{(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2}, \quad q = p\sin\phi + c\cos\phi \tag{9}$$

Eq. (1)[1] is satisfied automatically and Eq. (1)[2] transforms to

$$\sigma_{\varphi\varphi} = -p + q \tag{10}$$

Substituting Eqs. (8)–(10) into (2) gives

$$\begin{aligned} -\frac{(1-\sin\phi\cos 2\psi)}{\sin\phi}\frac{\partial \ln q}{\partial \ln r} + \sin 2\psi\frac{\partial \ln q}{\partial\theta} + 2\cos 2\psi\frac{\mathrm{d}\psi}{\mathrm{d}\theta} + 3\cos 2\psi - 1 + \sin 2\psi\cot\theta = 0 \\ \sin 2\psi\frac{\partial \ln q}{\partial \ln r} - \frac{(1+\sin\phi\cos 2\psi)}{\sin\phi}\frac{\partial \ln q}{\partial\theta} + 2\sin 2\psi\frac{\mathrm{d}\psi}{\mathrm{d}\theta} + 3\sin 2\psi - (1+\cos 2\psi)\cot\theta = 0 \end{aligned} \tag{11}$$

These equations are compatible if

$$\ln\frac{q}{c} = A\ln\frac{r}{R} + Q(\psi) \tag{12}$$

where A and R are constant. Substituting (12) into (11) it is possible, after some algebra, to obtain the following equations for ψ and Q:

$$\begin{aligned} &2\sin\phi(\sin\phi + \cos 2\psi)\frac{\mathrm{d}\psi}{\mathrm{d}\theta} - A\cos^2\phi + \sin\phi\sin 2\psi(1-\sin\phi)\cot\theta \\ &\quad + \sin\phi(3\cos 2\psi - 1 - \sin\phi\cos 2\psi + 3\sin\phi) = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{\mathrm{d}Q}{\mathrm{d}\psi}[A\cos\phi\cot\phi - \sin 2\psi(1-\sin\phi)\cot\theta - (3\cos 2\psi - 1 - \sin\phi\cos 2\psi + 3\sin\phi)] - 2A\sin 2\psi \\ &\quad - 2\sin\phi\sin 2\psi + 2\sin\phi(1+\cos 2\psi)\cot\theta = 0 \end{aligned} \tag{14}$$

Eq. (13) should be solved numerically with the boundary condition (5). It is seen from (13) that the coefficient of the derivative vanishes at $\psi = \psi_w$. The solution significantly depends on whether or not the derivative $\mathrm{d}\psi/\mathrm{d}\theta$ reduces to the expression 0/0 at $\psi = \psi_w$. It follows from (13) that the latter is possible if and only if

$$A = A_1 = \tan\phi(1-\sin\phi)\cot\theta_0 - \sin\phi \tag{15}$$

The special case $A = A_1$ is excluded from consideration. At arbitrary A not equal to A_1 expanding the coefficients of (13) in a Taylor series in a vicinity of the point $\psi = \psi_w$ and $\theta = \theta_0$ results in

$$(\psi - \psi_w)\frac{\mathrm{d}\psi}{\mathrm{d}\theta} = \frac{C}{2} \tag{16}$$

where

$$C = \frac{1}{2}[\cot\theta_0(1-\sin\phi) - \cot\phi(A+\sin\phi)] \tag{17}$$

Therefore, in a vicinity of the point $\theta = \theta_0$ the solution to Eq. (13) can be represented in the following form:

$$\psi = \psi_w - \sqrt{C}\sqrt{\theta - \theta_0} \tag{18}$$

It follows from this equation that $C \geqslant 0$ and, then, from (17) that $A \leqslant A_1$.

Because of the nature of the problem, there is no natural boundary condition for Eq. (14). However, in general $\sigma_{\theta\theta}$ must be negative on the friction surface. Therefore, using (5) it is possible to find from (8) that

$$\frac{q}{c} > \frac{1}{\cos\phi} \tag{19}$$

at $\theta = \theta_0$. Obviously, the condition (19) is not satisfied on the entire friction surface, unless $A = 0$. For, it follows from (12) that $q \to 0$ as $r \to \infty$ if $A < 0$ and $q \to 0$ as $r \to 0$ if $A > 0$. It is a typical drawback of this kind of solutions. For instant, it has been mentioned in Alexandrov and Goldstein (1993) that the same situation appears in the case of flow through infinite converging channels, for example in the solution given in Shield (1955). Moreover, for any value of the characteristic radius R it is possible to choose the constant of integration in the solution of Eq. (14) such that the condition (19) is satisfied for $0 \leqslant r \leqslant R$, if $A < 0$. The latter condition is compatible with the aforementioned condition $A < A_1$.

4. Velocity solution

The boundary conditions (6) and (7) require that the velocity u_θ is independent of r at $\theta = \theta_0$ and $\theta = \theta_p$, and that the velocity u_r is independent of r at $\theta = \theta_0$. Therefore, it is reasonable to assume that the velocity components are independent of r. Then, the incompressibility equation (3) transforms to

$$u_r = -\frac{1}{2}\left(\frac{du_\theta}{d\theta} + u_\theta \cot\theta\right) \tag{20}$$

It is convenient to use ψ as the independent variable instead of θ. Then, in particular,

$$\frac{du_\theta}{d\theta} = \frac{du_\theta}{d\psi}\frac{d\psi}{d\theta} \quad \text{and} \quad \frac{d^2u_\theta}{d\theta^2} = \frac{d^2u_\theta}{d\psi^2}\left(\frac{d\psi}{d\theta}\right)^2 + \frac{du_\theta}{d\psi}\frac{d^2\psi}{d\theta^2} \tag{21}$$

where the derivatives $d\psi/d\theta$ and $d^2\psi/d\theta^2$ are the known functions of ψ due to Eq. (13) and its solution. Using (20) and (21) the boundary condition (7)[1] can be rewritten in the form

$$\frac{du_\theta}{d\psi}\frac{d\psi}{d\theta} + u_\theta \cot\theta_p = -2U\cos\theta_p \tag{22}$$

at $\psi = \psi_p$ where ψ_p is the value of ψ at $\theta = \theta_p$. Substituting (20) into (4), with the use of (21), gives the following homogeneous linear second-order ordinary differential equation for u_θ

$$c_2\frac{d^2u_\theta}{d\psi^2} + c_1\frac{du_\theta}{d\psi} + c_0u_\theta = 0 \tag{23}$$

where

$$\begin{aligned}
c_2 &= \frac{B_1^2(\cos 2\psi + \sin\phi)}{8\sin^2\phi} \\
c_1 &= \frac{B_1\cot\theta(\cos 2\psi + \sin\phi)^2}{4\sin\phi} + \frac{B_2}{4\sin^2\phi\sin^2\theta} - \frac{B_1\sin 2\psi(\cos 2\psi + \sin\phi)}{4\sin\phi} \\
c_0 &= \frac{\cot\theta\sin 2\psi(\cos 2\psi + \sin\phi)^2}{2} - B_1(\cos 2\psi + \sin\phi) - \frac{\cos 2\theta(\cos 2\psi + \sin\phi)^3}{2\sin^2\theta} \\
B_1 &= A\cos^2\phi - \sin\phi\sin 2\psi(1 - \sin\phi)\cot\theta - \sin\phi(3\cos 2\psi - 1 - \sin\phi\cos 2\psi + 3\sin\phi) \\
B_2 &= \sin^2\phi(1 - \sin\phi)\sin 2\psi(\cos 2\psi + \sin\phi)^2 + B_1^2\sin 2\psi\sin^2\theta \\
&\quad - B_1\sin\phi\sin^2\theta(\cos 2\psi + \sin\phi)[(1 - \sin\phi)\cos 2\psi\cot\theta + (\sin\phi - 3)\sin 2\psi]
\end{aligned} \tag{24}$$

Here θ should be excluded by means of the solution to Eq. (13). It follows from (23) and (24) that $\psi = \psi_w$ is a regular singular point of Eq. (23). Expanding the coefficients of Eq. (23) in a Taylor series in a vicinity of the singular point gives

$$\begin{aligned} c_2 &= \frac{\cos^3\phi[\cot\theta_0\sin\phi(\sin\phi-1)+\cos\phi(A+\sin\phi)]^2}{4\sin^2\phi}(\psi-\psi_w) \\ c_1 &= \frac{\cos^3\phi[\cot\theta_0\sin\phi(\sin\phi-1)+\cos\phi(A+\sin\phi)]^2}{4\sin^2\phi} \\ c_0 &= -2\cos^2\phi[\cot\theta_0\sin\phi(\sin\phi-1)+\cos\phi(A+\sin\phi)](\psi-\psi_w) \end{aligned} \tag{25}$$

to leading order. Substituting (25) into (23) it is possible to arrive at the indicial equation in the form

$$\alpha(\alpha-1)-\alpha=0 \tag{26}$$

Then, one of the linearly independent primitive solutions of Eq. (23) is

$$u_1 = \sum_{n=0}^{\infty} a_n(\psi_w-\psi)^{n+2}, \quad a_0 \neq 0 \tag{27}$$

The second primitive solution can be found by means of a standard procedure with the use of the solution (27) to give

$$u_2 = \sum_{n=0}^{\infty} b_n(\psi_w-\psi)^{n} + \ln(\psi_w-\psi)\sum_{n=0}^{\infty} g_n(\psi_w-\psi)^{n+2}, \quad g_0 \neq 0 \tag{28}$$

The general solution to Eq. (23) is

$$u_\theta = C_1u_1(\psi)+C_2u_2(\psi) \tag{29}$$

It is possible to show that it is necessary to put $C_2 = 0$. The shear strain rate $\xi_{r\theta}$ contains the term $\mathrm{d}u_r/\mathrm{d}\theta$ or, according to (20), $\mathrm{d}^2u_\theta/\mathrm{d}\theta^2$. Therefore, Eqs. (16) and (21) show that the contribution of the solution (28) to the value of $\xi_{r\theta}$ in a vicinity of the surface $\psi = \psi_w$ contains, for example, a term of order

$$\xi_{r\theta} = \frac{E}{\theta_0-\theta}+\cdots \tag{30}$$

where E is independent of θ. Since the shear stress $\sigma_{r\theta} = \mathrm{O}(1)$, it follows from (30) that the work rate, W, contains a term of order $\mathrm{O}[(\theta_0-\theta)^{-1}]$ and, therefore,

$$\int_{\theta_0}^{\theta_p} W\,\mathrm{d}\theta \to \infty \tag{31}$$

showing that the solution (28) has no physical sense. Thus it is necessary to put $C_2 = 0$ in (29) such that $u_\theta = C_1u_1$. It is obvious that the solution (27) satisfies the boundary condition (6) at any a_0. The value of a_0 and θ_p can be found from the boundary conditions $(7)^2$ and (22) at any given A satisfying the condition $A < A_1$.

In order to perform calculations near the singular point, substitute the first two terms of the solution (27) and the representation (25) into (23). Collecting the coefficients of like powers of $(\psi_w - \psi)$ gives

$$a_1 = \frac{4\tan\phi[\sin 2\phi-(4-A)\cos\phi-2\cot\theta_0\sin\phi(1-\sin\phi)]}{3[\cos\phi(A+\sin\phi)-\cot\theta_0\sin\phi(1-\sin\phi)]}a_0 \tag{32}$$

In a narrow layer $\psi_w \leqslant \psi \leqslant \psi_\delta = (1-\delta)\psi_w$, where $\delta \ll 1$, the solution is approximated by the first two terms of (27) where a_1 is excluded by means of (32). In the interval $\psi_\delta \leqslant \psi \leqslant \psi_p$ the solution should be found numerically, guessing the value of a_0. Then, an iterative procedure should be used to find a_0 and θ_p from the boundary conditions at $\theta = \theta_p$. Note that θ_p is the known function of ψ_p due to the solution to Eq. (13).

The value of the strain rate intensity factor introduced in Alexandrov and Richmond (2001) may be of some interest for applications. This factor, D, has been defined as the coefficient of the singular term in the expansion

$$\xi_{\text{eq}} = \frac{D}{\sqrt{s}} + \text{o}\left(\frac{1}{\sqrt{s}}\right) \tag{33}$$

where $\xi_{\text{eq}} = \sqrt{(2/3)\xi_{ij}\xi_{ij}}$ is the equivalent strain rate, ξ_{ij} are the components of the strain rate tensor and s is the distance from the friction surface. In the case under consideration $s = r(\theta_0 - \theta)$. Since $\xi_{r\theta} \to \infty$ and the other strain rate components are finite as $\psi \to \psi_w$,

$$\xi_{\text{eq}} \approx \frac{2}{\sqrt{3}}\xi_{r\theta} \tag{34}$$

as $\psi \to \psi_w$. Using (27), (16) and (32) the shear strain rate can be calculated and then substituted into (34) to show that the equivalent strain rate follows the inverse square root rule (33) in the vicinity of the friction surface and to find the strain rate intensity factor in the form

$$D = \frac{D_1}{\sqrt{r}} \tag{35}$$

where

$$D_1 = \frac{\sqrt{3}|a_1|C\sqrt{C}}{8} \tag{36}$$

Here a_1 should be excluded by means of (32). The rule (33) is also valid in plane-strain solutions based on the double-shearing model (Pemberton, 1965; Marshall, 1967; Alexandrov and Lyamina, 2002, 2003).

5. Plastic work rate

It is necessary to check that the plastic work rate is positive, $\sigma_{ij}\xi_{ij} > 0$. Using (8) and expressing the strain rate components through the velocity components leads to the following inequality:

$$\Omega = (1 + \cos 2\psi)\left(\frac{\mathrm{d}u_\theta}{\mathrm{d}\theta} - u_\theta \cot\theta\right) + \sin 2\psi\left(\frac{\mathrm{d}^2 u_\theta}{\mathrm{d}\theta^2} + \frac{\mathrm{d}u_\theta}{\mathrm{d}\theta}\cot\theta - u_\theta \cot^2\theta\right) < 0 \tag{37}$$

This inequality can be checked numerically once the solution to Eqs. (13) and (23) has been found.

6. Numerical example

With no loss of generality, it is possible to put $U = 1$. A typical angle of internal friction is $\phi = \pi/6$. It is also assumed that $\theta_0 = \pi/6$. Then, it follows from (15) that $A_1 = 0$ and from (19) that the minimum value of q/c at the friction surface is $2/\sqrt{3}$. Therefore, the solution is valid at $A < 0$. To satisfy the condition that $\sigma_{\theta\theta} < 0$ on the friction surface at $r \leqslant R$, it is necessary to put $Q > \ln(2/\sqrt{3})$ at $\psi = \psi_w$, as follows from (12). Fig. 3 shows the dependence of the orientation of the major principal stress in a meridian plane on θ at different A obtained from Eq. (13). To illustrate the variation of the stress components with θ, Eq. (14) has been solved with the boundary condition $Q = \ln 3$ at $\psi = \psi_w$. Then, the stress components have

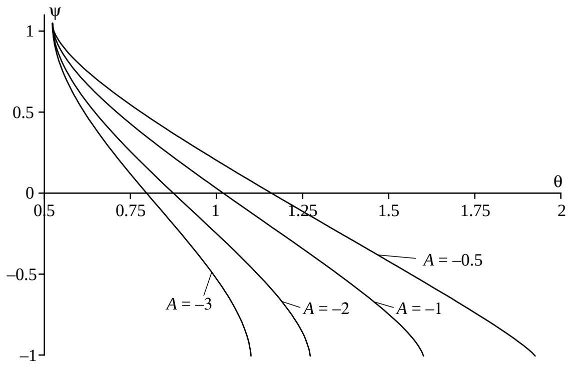

Fig. 3. Dependence of ψ on θ at different A.

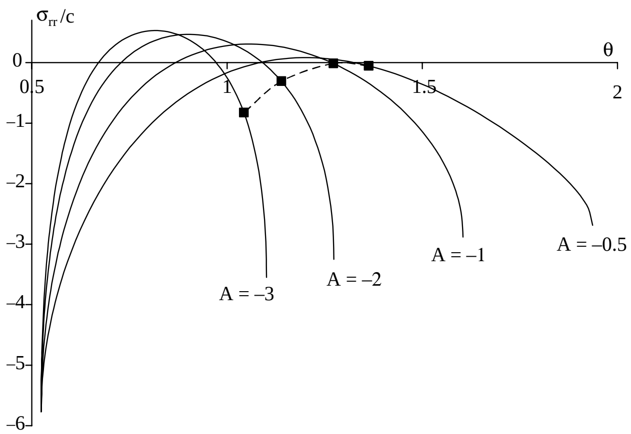

Fig. 4. Dependence of the stress σ_{rr} on θ at $r = R$.

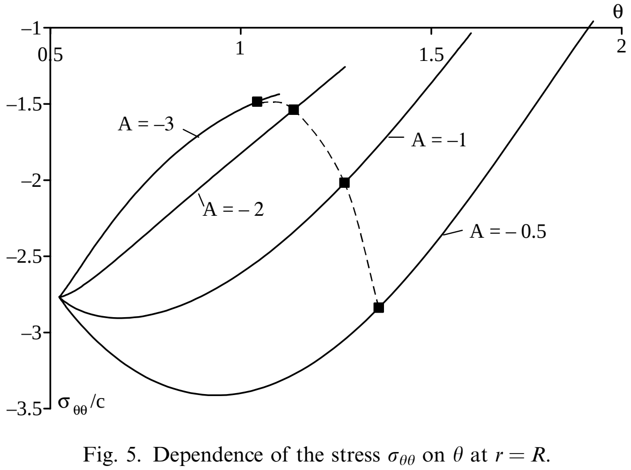

Fig. 5. Dependence of the stress $\sigma_{\theta\theta}$ on θ at $r = R$.

been calculated by means of (8)–(10) and (12). The dependence of the stress components on θ at $r = R$ is depicted in Figs. 4–7 (solid lines). Having the ψ distribution, Eq. (23) has been solved for u_θ and, then, u_r has been found by means of (20). The solution to Eq. (23) also determines the orientation of the rigid/plastic boundary, θ_p, and, with the use of (35) and (36), the strain rate intensity factor. The variation of θ_p with A is depicted in Fig. 8. The dashed lines in Figs. 4–7 correspond to the rigid plastic boundary. Note that even though the stress solution is extended into the plastic zone without violating the equilibrium equations and the yield condition, it is not extended over the entire rigid zone. Therefore, the solution obtained is not complete in the sense that it is unknown if a statically admissible stress field exists in the entire rigid zone. Such a drawback is typical in solutions of this kind (for example, Durban and Fleck, 1992). The dependence of the strain rate intensity factor on r is obvious from (35). Therefore, Fig. 9 shows the variation of D_1 with A. The dependence of the velocity components with θ within the plastic zone at different A is presented in Figs. 10 and 11. Finally, the value of Ω has been calculated according to (37). The calculation has demonstrated that the plastic work rate is positive in the case considered.

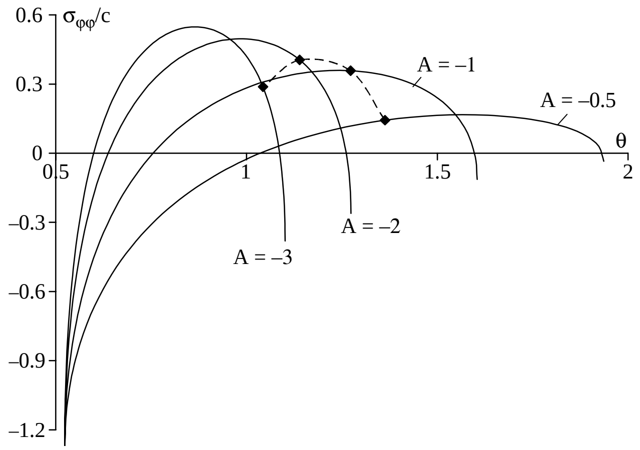

Fig. 6. Dependence of the stress $\sigma_{\varphi\varphi}$ on θ at $r = R$.

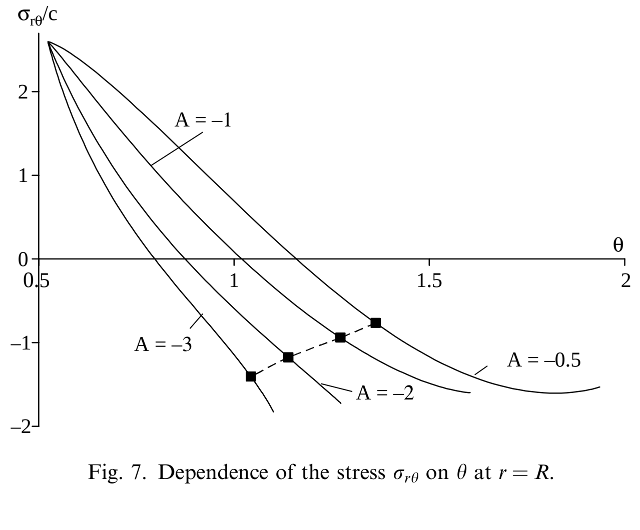

Fig. 7. Dependence of the stress $\sigma_{r\theta}$ on θ at $r = R$.

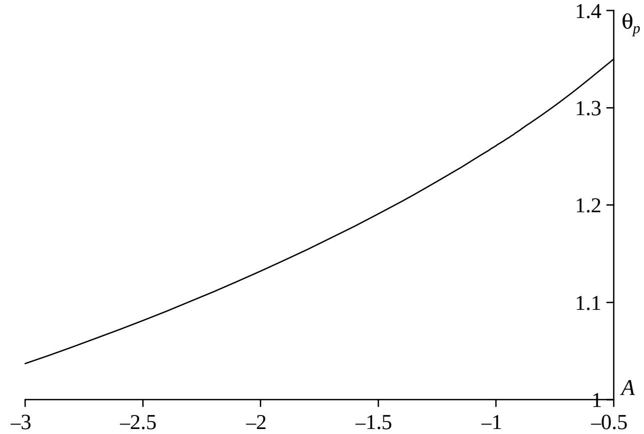

Fig. 8. Variation of angle θ_p determining the orientation of the rigid/plastic boundary with A.

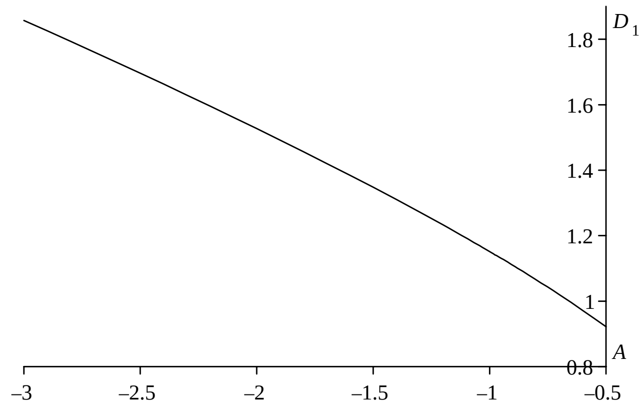

Fig. 9. Variation of D_1 determining the magnitude of the strain rate intensity factor with A.

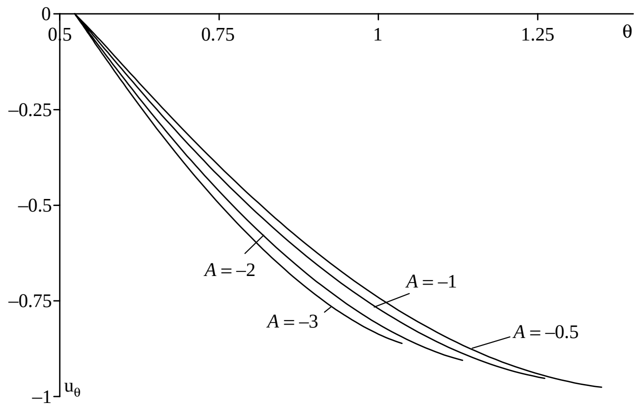

Fig. 10. Dependence of the velocity component u_θ on θ within the plastic zone at different A.

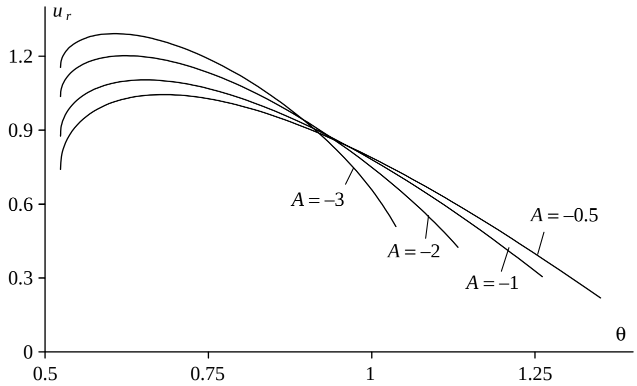

Fig. 11. Dependence of the velocity component u_r on θ within the plastic zone at different A.

7. Conclusions

A new semi-analytical solution has been obtained for axisymmetric penetration of a rigid cone into plastic medium obeying the double-shearing model. As in other similar problems, not all constants are determined by the boundary conditions and the normal stress is not compressive on a part of the friction surface. The solution develops the same type of singularity in the vicinity of the maximum friction surface as plane-strain solutions based on the double-shearing model (Alexandrov and Lyamina, 2002) and solutions of classical plasticity (Alexandrov and Richmond, 2001). In particular, in the case of steady penetration of a rigid cone into perfectly plastic materials the singular velocity fields have been found by Fleck and Durban (1991) and Durban and Fleck (1992). Nevertheless, it is important to mention that not all models of pressure-dependent plasticity lead to singular solutions of the type obtained (Alexandrov, 2003). Also, using other constitutive laws it is possible to arrive at quite different behavior of solutions near maximum friction surfaces. An example is given in Fleck and Durban (1991) where, in the case of power-law viscous solids, sticking has been obtained at the maximum friction surface. Some general results on the solution behavior of rate-dependent and hardening, pressure-independent plastic materials near maximum friction surfaces are given in Alexandrov and Alexandrova (2000a,b) and Alexandrov et al. (2000).

Numerical results illustrate the variation of stress and velocity with the angular coordinate. An asymptotic analysis of the solution near the friction surface has been used to solve the differential equation (23) and to extract the strain rate intensity factor.

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